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2004 J. Phys. A: Math. Gen. 37 511

(http://iopscience.iop.org/0305-4470/37/2/018)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 511–520 PII: S0305-4470(04)63433-0

The dilute A_L models and the integrable perturbations of unitary minimal CFTs

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Received 14 May 2003
Published 15 December 2003
Online at stacks.iop.org/JPhysA/37/511 (DOI: 10.1088/0305-4470/37/2/018)

Abstract

Recently, a set of thermodynamic Bethe ansatz (TBA) equations has been proposed by Dorey, Pocklington and Tateo for unitary minimal models perturbed by the $\phi_{1,2}$ or $\phi_{2,1}$ operator. We examine their results in view of the lattice analogues, dilute A_L models in regimes 1 and 2. Taking $M_{5,6} + \phi_{1,2}$ and $M_{3,4} + \phi_{2,1}$ as the simplest examples, we will explicitly show that the conjectured TBA equations can be recovered from the lattice model in a scaling limit.

PACS numbers: 02.30.Ik, 05.50.+q

1. Introduction

Since the breakthrough in the integrable perturbation theory of CFT [1, 2], there has been a lot of progress in the understanding of $\phi_{1,3}$ perturbation theory [3, 4]. On the other hand, although the remarkable example, the Ising model in a magnetic field, was treated in [1], the progress on the $\phi_{1,2}$ and $\phi_{2,1}$ perturbed theories has been steady but slow.

Systematic studies on the bootstrap procedure on the S matrix were initiated in [5] and [6]. The latter approach, based on the scaling q-state Potts field theory, has been further elaborated by Dorey *et al* [7]. Thanks to the Coleman–Thun mechanism, they argue that the contributions from spurious poles cancel and conclude the closed set of S-matrices for a wide range of parameters.

The check of the results against a finite-size system, however, suffers from the non-diagonal nature of the scattering process. Due to the lack of a relevant string hypothesis, the diagonalization of the transfer matrix is far from trivial. In [8], a set of thermodynamic Bethe ansatz (TBA) equations is conjectured from consideration of special cases for which they found similarity to the TBA for the sine–Gordon model. Roughly speaking, they proposed the TBA by gluing the 'breather–kink' part and the 'magnon' part in which the latter originates from the sine-Gordon model at specific coupling [9, 10]. Although the derivation is intuitive, the resultant equations pass many non-trivial checks.

In this report, we shall examine the problem in view of a solvable lattice model. As a lattice analogue to $M_{L,L+1} + \phi_{1,2}$, $M_{L+1,L+2} + \phi_{2,1}$ we consider the L-state RSOS model proposed in [11, 12], which will be referred to as the dilute A_L model. There are several pieces of evidence for this correspondence, the central charge [11], the scaling dimensions of the leading perturbation [11, 13], universal ratios [14–16] and so on.

The question whether it shares the identical TBA to describe its finite-temperature (size) property has not yet been fully answered. The purpose of this report is to present positive evidence for this inquiry.

There are already a few examples demonstrating the equivalence. The common TBA of the dilute A_3 model at regime 2 and the $M_{3,4} + \phi_{1,2}$ case was first proved in [17]. The most dominant solutions to the Bethe ansatz equation are explicitly identified in the form of the 'string solution', which leads to the famous E_8 TBA. In the case $L = 4, 6$, corresponding to the E_7, E_6 case, such explicit identification of string hypothesis seems not yet to be completed.

An alternative approach, based on the quantum transfer matrix (QTM) [18, 19], has been successfully applied to $L = 3, 4, 6$ [20, 21]. The functional relations among properly chosen QTMs play a fundamental role there and it enables to TBA to be derived without knowing the explicit locations of dominant solutions to the Bethe ansatz equation.

For $L = 3, 4, 6$ cases, the underlying affine Lie algebraic structure (E_8, E_7, E_6, respectively) provides several clues in the investigation of the functional relations among QTMs. The remaining case seems to lose a direct connection to affine Lie algebra in general (see, however, exceptions [8]). It might thus be challenging to clarify the functional relation, and thereby see if the Y-system in is actually recovered. In this report, we focus on the last 'exceptional' case (in the terminology of [8]) $M_{5,6}$ for the $\phi_{1,2}$ perturbation, and the first exceptional case $M_{3,4}$ for the $\phi_{2,1}$ perturbation.

This paper is organized as follows. In the next section, we give a brief review of the dilute A_L models and the QTM method. Section 3 is devoted to the discussion of the dilute A_5 model at regime 2 which is expected to be a lattice analogue of the $M_{5,6} + \phi_{1,2}$ theory. Fusion QTMs parametrized by skew Young diagrams are introduced and found to satisfy a set of closed functional relations. It will be shown that the conjectured TBA is naturally derived in a scaling limit. In the case of the dilute A_L model, L even, a fundamental role seems to be played by a 'kink' transfer matrix. As the simplest and the most well-known example, we treat $M_{3,4} + \phi_{2,1}$, corresponding to the Ising model off critical temperature, in section 4. We conclude the paper with a brief summary and discussion in section 5.

2. The dilute A_L model and the quantum transfer matrix

The dilute A_L model is proposed in [11] as an elliptic extension of the Izergin–Korepin model [22]. The model is of the restricted SOS type with local variables $\in \{1, 2, \ldots, L\}$. The variables $\{a, b\}$ on neighbouring sites should satisfy the adjacency condition, $|a-b| \leqslant 1$, which is often described by a graph in figure 1. In [11], the RSOS weights, satisfying the Yang–Baxter relation, have been found to be parametrized by the spectral parameter u and the elliptic nome q. The crossing parameter λ needs to be a function of L for the restriction. The model exhibits four different physical regimes depending on parameters:

- regime 1. $0 < u < 3$ $\quad \lambda = \frac{\pi L}{4(L+1)}$ $\quad L \geqslant 2$
- regime 2. $0 < u < 3$ $\quad \lambda = \frac{\pi(L+2)}{4(L+1)}$ $\quad L \geqslant 3$
- regime 3. $3 - \frac{\pi}{\lambda} < u < 0$ $\quad \lambda = \frac{\pi(L+2)}{4(L+1)}$ $\quad L \geqslant 3$
- regime 4. $3 - \frac{\pi}{\lambda} < u < 0$ $\quad \lambda = \frac{\pi L}{4(L+1)}$ $\quad L \geqslant 2.$

We are interested in regimes 1 and 2.

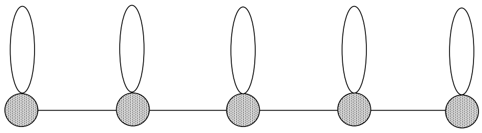

Figure 1. An incidence diagram for the dilute A_5 model. The local states corresponding to connected nodes can be located at the nearest neighbour sites on a square lattice.

The central charge and scaling dimension associated with the leading perturbation evaluated in [11, 13] suggests

- The dilute A_{L-1} model in regime 1 is an underlying lattice theory for $M_{L,L+1} + \phi_{2,1}$.
- The dilute A_L model in regime 2 is an underlying lattice theory for $M_{L,L+1} + \phi_{1,2}$.

There is also further evidence supporting this correspondence, as mentioned in the introduction.

One can introduce an associated 1D quantum system with the above 2D classical model. The Hamiltonian $\mathcal{H}_{1\mathrm{D}}$ for the former is defined from the row to row transfer matrix $T_{\mathrm{RTR}}(u)$ of the latter, by

$$\mathcal{H}_{1\mathrm{D}} = \epsilon \frac{\partial}{\partial u} \log T_{\mathrm{RTR}}(u)|_{u=0}.$$

We omit the explicit operator form of $\mathcal{H}_{1\mathrm{D}}$. The parameter $\epsilon = -1\,(1)$ labels regimes 1 and 2 (3 and 4).

The thermodynamics of the 1D quantum system is the central issue in the following. We apply the method of QTM [18, 19] to this problem. Leaving details to the references, we list the only relevant results for the following discussion.

A fundamental QTM is defined in a staggered manner

$$(T_{\mathrm{QTM}}(u,x))_{\{a\}}^{\{b\}} = \prod_{j=1}^{N/2} \begin{array}{ccc} b_{2j-1} & & b_{2j} \\ & \boxed{u+\mathrm{i}x} & \\ a_{2j-1} & & a_{2j} \end{array} \begin{array}{ccc} a_{2j+1} & & b_{2j+1} \\ & \boxed{u-\mathrm{i}x} & \\ a_{2j} & & b_{2j} \end{array}.$$

In the above, squares represent Boltzmann weights; four indices represent local variables and the spectral parameters are specified inside them. The fictitious dimension N(even), sometimes referred to as the Trotter number, is introduced. It has nothing to do with the real system size of the original 1D system. The real system size will not appear in our discussion as the quantities after taking the thermodynamic limit are of interest to us.

It is vital that two (spectral) parameters u, x exist and that only the latter concerns the commutative property of QTMs, $[T_{\mathrm{QTM}}(u,x), T_{\mathrm{QTM}}(u,x')] = 0$. The remaining parameter u plays the role of intertwining the finite Trotter number (N) system and the finite-temperature system (β) by $u = u^* = -\epsilon\frac{\beta}{N}$. More concretely, the free energy per site is represented *only* by the largest eigenvalue $T_1(u,x)$ of T_{QTM} at $x = 0$ and $u = u^*$,

$$\beta f = -\lim_{N\to\infty} \log T_1(u^*, x=0).$$

The eigenvalue $T_1(u,x)$ takes the form

$$\begin{aligned} T_1(u,x) &= w\phi\left(x+\frac{3}{2}\mathrm{i}\right)\phi\left(x+\frac{1}{2}\mathrm{i}\right)\frac{Q\left(x-\frac{5}{2}\mathrm{i}\right)}{Q\left(x-\frac{1}{2}\mathrm{i}\right)} \\ &\quad +\phi\left(x+\frac{3}{2}\mathrm{i}\right)\phi\left(x-\frac{3}{2}\mathrm{i}\right)\frac{Q\left(x-\frac{3}{2}\mathrm{i}\right)Q\left(x+\frac{3}{2}\mathrm{i}\right)}{Q\left(x-\frac{1}{2}\mathrm{i}\right)Q\left(x+\frac{1}{2}\mathrm{i}\right)} \\ &\quad +w^{-1}\phi\left(x-\frac{3}{2}\mathrm{i}\right)\phi\left(x-\frac{1}{2}\mathrm{i}\right)\frac{Q\left(x+\frac{5}{2}\mathrm{i}\right)}{Q\left(x+\frac{1}{2}\mathrm{i}\right)} \end{aligned}$$

$$Q(x) := \prod_{j=1}^{N} h[x - x_j]$$

$$\phi(x) := \left(\frac{h\left[x + \left(\frac{3}{2} - u\right) \mathrm{i}\right] h\left[x - \left(\frac{3}{2} - u\right) \mathrm{i}\right]}{h[2\mathrm{i}]h[3\mathrm{i}]} \right)^{N/2} \qquad h[x] := \vartheta_1(\mathrm{i}\lambda x) \tag{1}$$

where $w = \exp\left(\mathrm{i}\frac{\pi\ell}{L+1}\right)$ ($\ell = 1$ for the largest eigenvalue sector).

The parameters $\{x_j\}$ are solutions to the 'Bethe ansatz equation' (BAE),

$$w\frac{\phi(x_j + \mathrm{i})}{\phi(x_j - \mathrm{i})} = -\frac{Q(x_j - \mathrm{i})Q(x_j + 2\mathrm{i})}{Q(x_j + \mathrm{i})Q(x_j - 2\mathrm{i})} \qquad j = 1, \ldots, N. \tag{2}$$

From now on we suppress the dependence on u which must be set as $u = u^*$.

It has been shown in many examples [23], that the functional relations among 'generalized' (fusion) QTMs offer a way to evaluate the free energy without precise knowledge of the locations $\{x_j\}$. We adopt the same strategy here and shall discuss the functional relations realized among fusion QTMs of the dilute A_L model below.

3. QTM associated with skew Young diagrams and quantum Jacobi–Trudi formula

We introduce fusion QTMs associated with Young diagrams. The idea of connecting Young diagrams and (eigenvalues of) QTM, originated in [24–26], is very simple. Let three boxes with letters 1, 2 and 3 represent the three terms in the eigenvalue of the QTM (1),

$$T_1(x) = \boxed{1}_x + \boxed{2}_x + \boxed{3}_x.$$

Obviously, the eigenvalue of a fusion QTM can be represented by a summation of products of 'boxes' with different letters and spectral parameters, over a certain set. The point is that the set can be identified with semi-standard Young tableaux (SST) for sl_3. We state the above situation more precisely. Let μ and λ be a pair of Young tableaux satisfying $\mu_i \geqslant \lambda_i, \forall i$. We subtract a diagram λ from μ, which is called a skew Young diagram $\mu - \lambda$. The usual Young diagram is the special case that λ is empty, and we will omit λ in the case hereafter. On each diagram, the spectral parameter changes $+2i$ from the left box to the right and $-2i$ from the top box to the bottom. We fix the spectral parameter associated with the right-top box to be $x + \mathrm{i}(\mu'_1 + \mu_1 - 2)$ (or equivalently the spectral parameter associated with the left-bottom box to be $x - \mathrm{i}(\mu'_1 + \mu_1 - 2)$). Insert a letter $\ell_{i,j}$ to the (i, j)-th box such that the semi-standard condition is satisfied. We denote its spectral parameter by $x_{i,j}$. Then the product

$$\prod_{i,j} \boxed{\ell_{i,j}}_{x_{i,j}}$$

is associated with the Young table. The summation over the tableaux satisfying the semi-standard condition then defines

$$\mathcal{T}^{\vee}_{\mu/\lambda}(x) = \sum_{\{\ell_{i,j}\}\in\mathrm{SST}} \prod_{i,j} \boxed{\ell_{i,j}}_{x_{i,j}} \tag{3}$$

which is expected to be the eigenvalue of a fusion QTM.

The simplest subset of the above is the QTM based on Young diagrams of the rectangular shape. It was shown [20] that any such QTM is proportional to a QTM corresponding to a Young diagram of a shape $1 \times m$. The latter is associated to m-fold symmetric fusion. For later convenience, we introduce a renormalized $1 \times m$ fusion QTM $T_m(x)$ by

$$T_m(x) = \frac{1}{f_m(x)} \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m} \boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_m}.$$

The renormalization factor f_m, common to tableaux of width m, is given by

$$f_m(x) := \prod_{j=1}^{m-1} \phi\left(x \pm \mathrm{i}\left(\frac{2m-1}{2} - j\right)\right).$$

Hereafter, for any function $f(x)$, we denote by $f(x \pm \mathrm{i}a)$ the product $f(x + \mathrm{i}a)f(x - \mathrm{i}a)$.

Then the resultant T_m are all degree $2N$ wrt $h[x + \text{shift}]$, and have a periodicity due to Boltzmann weights; $T_m(x + P\mathrm{i}) = T_m(x)$, where

$$P = \begin{cases} \frac{4(L+1)}{L+2} & \text{for regime 2} \\ \frac{4(L+1)}{L} & \text{for regime 1.} \end{cases} \tag{4}$$

Remarkably, $T_m(x)$ enjoys a 'duality'

$$T_m(x) = \begin{cases} T_{2L-1-m}(x) & m = 0, \ldots, 2L \quad \text{for } L \text{ even} \\ T_{2L-1-m}\left(x + \frac{P}{2}\mathrm{i}\right) & m = 0, \ldots, 2L \quad \text{for } L \text{ odd .} \end{cases} \tag{5}$$

This is deduced from the $a_2^{(2)}$ nature of the model and special choice of λ. We have at least checked the validity numerically and assume their validity in this report. The above two properties, the periodicity and the duality (5), play a fundamental role in the proof of the closed functional relations.

The real usefulness of $T_m(x)$ lies in the fact that any QTM associated with a skew Young diagram can be represented in terms of their products.

Theorem 1. *Let $\mathcal{T}_{\mu/\lambda}(x)$ be a renormalized $\mathcal{T}_{\mu/\lambda}(x)$ in (3) by a common factor,* $\prod_{j=1}^{\mu_1'} f_{\mu_j-\lambda_j}$ $(x + \mathrm{i}(\mu_1' - \mu_1 + \mu_j + \lambda_j - 2j + 1))$. *Then the following equality holds.*

$$\mathcal{T}_{\mu/\lambda}(x) = \det_{1 \leqslant j,k \leqslant \mu_1'}(T_{\mu_j-\lambda_k-j+k}(x + \mathrm{i}(\mu_1' - \mu_1 + \mu_j + \lambda_k - j - k + 1))) \tag{6}$$

where $T_{m<0} := 0$.

We regard this as a quantum analogue of the Jacobi–Trudi formula.

By this, apparently $\mathcal{T}_{\mu/\lambda}(x)$ is an analytic function of x due to BAE, and contains the quantity of our interest, $T_1(x)$, as a special case. The former assertion is not obvious from the original definition by the tableaux, but it is trivial from the quantum Jacobi–Trudi formula.

In the same spirit, we introduce $\Lambda_{\mu/\lambda}(x)$, which is analytic under BAE,

$$\Lambda_{\mu/\lambda}(x) := \mathcal{T}_{\mu/\lambda}(x)/\{T_{m \geqslant 2L}(x) \to 0\}.$$

The pole-free property of $\Lambda_{\mu/\lambda}(x)$ is apparent from (6).

4. Dilute A_5 model in regime 2 as a lattice analogue to $M_{5,6} + \phi_{1,2}$

For $M_{5,6} + \phi_{1,2}$, Dorey *et al* argued the existence of two kinds of particles, two kinks and four breathers. For diagonalization of scattering theory, they introduced two magnons (massless particles) in addition. Explicitly, the Y-system reads

$$Y_{B_1}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = \Xi_{B_3}(x) \qquad Y_{B_3}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = \Xi_{B_1}(x)\Xi_{B_5}(x)$$

$$Y_{B_5}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = \Xi_{B_3}(x)\Xi_{K_2}\left(x \pm \tfrac{2}{14}\mathrm{i}\right)\Xi_{K_1}(x)\Xi_1\left(x \pm \tfrac{1}{14}\mathrm{i}\right)\Xi_2(x)$$

$$Y_{B_2}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = \Xi_{K_1}\left(x \pm \tfrac{2}{14}\mathrm{i}\right)\Xi_1\left(x \pm \tfrac{1}{14}\mathrm{i}\right)\Xi_{K_2}(x)$$

$$Y_{K_2}\left(x \pm \tfrac{1}{14}\mathrm{i}\right) = \Xi_{B_5}(x)\mathcal{L}^{(1)}(x) \qquad Y_{K_1}\left(x \pm \tfrac{1}{14}\mathrm{i}\right) = \Xi_{B_2}(x)\mathcal{L}^{(1)}(x)$$

$$Y_1\left(x \pm \tfrac{1}{14}\mathrm{i}\right) = \mathcal{L}_2(x)\mathcal{L}_{K_2}(x)\mathcal{L}_{K_1}(x) \qquad Y_2\left(x \pm \tfrac{1}{14}\mathrm{i}\right) = \mathcal{L}_1(x)$$

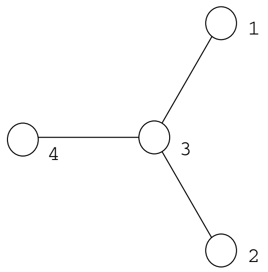

Figure 2. The nodes in the D_4 Dynkin diagram are indexed in the above manner.

with

$$\mathcal{L}_a(x) := \frac{1}{1 + \frac{1}{Y_a(x)}} \qquad \Xi_a(x) := 1 + Y_a(x)$$

where a takes one of $B_1, B_3, \ldots, 1, 2$. (Y_1, Y_2 are written as $Y^{(1)}, Y^{(2)}$ in [18].)

We are not starting from Y but rather from the QTM. Corresponding to breathers, we introduce 'breather' QTM by

$$\begin{aligned}
&T_{B_1}(x) := T_1(x)\\
&T_{B_3}(x) := \Lambda_{(8,1)}\left(x + \tfrac{13}{14}\mathrm{i}\right)\big/\phi\left(x - \tfrac{12}{7}\mathrm{i}\right)\\
&T_{B_5}(x) := \Lambda_{(15,8,8)/(7,7)}(x)/\phi\left(x \pm \tfrac{3}{2}\mathrm{i}\right)\\
&T_{B_7}(x) := \Lambda_{(15,15,8,8)/(14,7,7)}\left(x + \tfrac{11}{14}\mathrm{i}\right)\big/\left(\phi\left(x - \tfrac{12}{7}\mathrm{i}\right)\phi\left(x \pm \tfrac{9}{7}\mathrm{i}\right)\right)\\
&T_{B_2}(x) := T_7(x)\\
&T^{(6)}(x) := \Lambda_{(8,7)/(6)}\left(x + \tfrac{25}{14}\mathrm{i}\right)
\end{aligned}$$

then the following relations, referred to as the 'breather' T-system, hold

$$\begin{aligned}
&T_{B_1}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = T_0\left(x \pm \tfrac{11}{14}\mathrm{i}\right) + \phi\left(x - \tfrac{12}{7}\mathrm{i}\right)T_{B_3}(x)\\
&T_{B_3}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = T_0(x)T_0\left(x \pm \tfrac{8}{14}\mathrm{i}\right) + T_{B_1}(x)T_{B_5}(x)\\
&T_{B_5}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = T_0\left(x \pm \tfrac{3}{14}\mathrm{i}\right)T_0\left(x \pm \tfrac{5}{14}\mathrm{i}\right) + T_{B_3}(x)T_{B_7}(x)\\
&T_{B_2}\left(x \pm \tfrac{3}{14}\mathrm{i}\right) = T_0\left(x \pm \tfrac{1}{14}\mathrm{i}\right) + T^{(6)}(x)
\end{aligned}$$

where $T_0(x) = f_2(x)$. They originate from the 'hidden $su(2)$' discussed in [27].

In contrast to the dilute A_3 model (equivalently the E_8 case), the 'hidden $su(2)$' structure is not enough to obtain a closed set of functional relations. We then introduce another set of functional relations, related to magnons.

With each node on the D_4 Dynkin diagram (see figure 2), we associate $t_m^{(a)}(x)$ ($a = 1, 2, 3, 4$, $m \in Z_{\geqslant 0}$) and set $t_0^{(a)}(x) = 1$. Then we impose a D_4-related T-system among them, in the terminology of [28],

$$t_m^{(a)}\left(x \pm \frac{\mathrm{i}}{14}\right) = t_{m-1}^{(a)}(x)t_{m+1}^{(a)}(x) + g_m^{(a)}(x)\prod_{b\sim a} t_m^{(b)}(x) \tag{7}$$

where $g_1^{(a)}\left(x \pm \frac{\mathrm{i}}{14}\right) = g_2^{(a)}(x)$. In the above by $b \sim a$, we mean that a and b are connected on the Dynkin diagram.

Moreover we set an inhomogeneous truncation, $t_3^{(3)} = t_3^{(4)} = 0$ and put $g_1^{(3)} = g_1^{(4)} = 1$.

Unless one introduces some further condition, the set of functional relations (7) are not closed, so cannot be solved. Then we demand

$$t_3^{(1)}(x) = t_2^{(3)}(x) \qquad t_3^{(2)}(x) = t_2^{(3)}(x) T_{B_3}(x)$$
$$g_1^{(1)}(x) = T_0(x) \qquad g_1^{(2)}(x) = T_0\left(x \pm \frac{2\mathrm{i}}{7}\right).$$

The second relation glues the breather T-system to the D_4-related T-system.

The above requirements seem to be rather artificial, but they lead to remarkable consequences. First, solutions to (7) can be given in terms of QTM appearing in the dilute A_5 model as follows:

$$t_1^{(1)}(x) = T^{(6)}(x) \qquad t_1^{(2)}(x) = T_{B_7}(x)$$
$$t_1^{(3)}(x) = \Lambda_{(12,8,7)/(5,4)}(x) \qquad t_1^{(4)}(x) = \Lambda_{(5,1)}\left(x + \frac{15}{14}\mathrm{i}\right) \Big/ \phi\left(x - \frac{13}{2}\mathrm{i} + \frac{15}{14}\mathrm{i}\right)$$
$$t_2^{(1)}(x) = T_{B_5}(x) T_{B_2}\left(x \pm \frac{\mathrm{i}}{7}\right) \qquad t_2^{(2)}(x) = T_{B_5}\left(x \pm \frac{\mathrm{i}}{7}\right) T_{B_2}(x)$$
$$t_2^{(3)}(x) = T_{B_5}\left(x \pm \frac{\mathrm{i}}{14}\right) T_{B_2}\left(x \pm \frac{\mathrm{i}}{14}\right) \qquad t_2^{(4)}(x) = T_{B_5}(x) T_{B_2}(x).$$

The proof of the above statement is too lengthy to reproduce here. We hope to present it with the general discussion of L [29].

Second, the following combination of T and t solves the Y-system for $M_{5,6} + \phi_{1,2}$:

$$Y_{B_1}(x) = \frac{\phi\left(x - \frac{12}{7}\mathrm{i}\right) T_{B_3}(x)}{T_0\left(x \pm \frac{11}{14}\mathrm{i}\right)} \qquad Y_{B_3}(x) = \frac{T_{B_1}(x) T_{B_5}(x)}{T_0(x) T_0\left(x \pm \frac{8}{14}\mathrm{i}\right)}$$
$$Y_{B_5}(x) = \frac{T_{B_3}(x) T_{B_7}}{T_0\left(x \pm \frac{3}{14}\mathrm{i}\right) T_0\left(x \pm \frac{5}{14}\mathrm{i}\right)} \qquad Y_{B_2}(x) = \frac{T^{(6)}(x)}{T_0\left(x \pm \frac{1}{14}\mathrm{i}\right)}$$
$$Y_{K_1}(x) = \frac{t_2^{(1)}(x)}{t_1^{(3)}(x) g_1^{(1)}(x)} \qquad Y_{K_2}(x) = \frac{t_2^{(2)}(x)}{t_1^{(3)}(x) g_1^{(2)}(x)}$$
$$Y_1(x) = \frac{t_2^{(3)}(x)}{t_1^{(1)}(x) t_1^{(2)}(x) t_1^{(4)}(x)} \qquad Y_2(x) = \frac{t_2^{(4)}(x)}{t_1^{(3)}(x)}.$$

Third, the functions T, t, Y possess 'nice' analytic properties. Before stating the properties, we need preparations. Note that the Y-system is invariant, for even N, if Y is replaced by $\tilde{Y}$, defined by

$$\tilde{Y}_{B_1}(x) = \begin{cases} \dfrac{Y_{B_1}(x)}{\kappa\left(x \pm \mathrm{i}(1+u')\frac{3}{14}\right)} & \text{for } u < 0 \\ Y_{B_1}(x)\kappa\left(x \pm \mathrm{i}(1-u')\frac{3}{14}\right) & \text{for } u > 0 \end{cases}$$

and all other cases, $\tilde{Y}_a = Y_a$. The parameter u' stands for $\frac{14}{3}u$. This is due to the definition of κ,

$$\kappa(x) = \left(\mathrm{i}\frac{\vartheta_1\left(\mathrm{i}\frac{7}{6}\pi x, \tau'\right)}{\vartheta_2\left(\mathrm{i}\frac{7}{6}\pi x, \tau'\right)}\right)^N$$

which satisfies $\kappa\left(x \pm \mathrm{i}\frac{3}{14}\right) = 1$. The elliptic nome $q' = \exp(-\tau')$, $\tau' = 4\tau$ is introduced so as to respect the periodicity of the Y function on the real direction of x. We denote a typical $\tilde{Y}$ equation as

$$\tilde{Y}_a(x \pm \mathrm{i}\alpha) = \prod_b \Xi_b(x \pm \mathrm{i}\gamma_b) \prod_c \mathcal{L}_c(x \pm \mathrm{i}\gamma_c). \tag{8}$$

Our numerical data indicate that the RHS is analytic and nonzero in the strip $\mathrm{Im}\, x \in [-\alpha, \alpha]$. Each element in the LHS also satisfies the same in appropriate strips, i.e., $\Xi_b(x)$ is analytic and nonzero in the strip $\mathrm{Im}\, x \in [-\gamma_b, \gamma_b]$, and so on. These remarkable properties enable us to solve the coupled algebraic equation, such as (8), in the Fourier space (to be precise, its logarithmic derivatives). Then the inverse Fourier transformation leads to the coupled integral equations which yield the explicit evaluation of $\log Y_a(x)$.

To make a direct contact with the TBA result, three further steps are needed. First take the Trotter limit $N \to \infty$, $uN = \beta$ ($\epsilon = -1$). Second rewrite $\log \Xi_b(x)$ by $\log \mathcal{L}_b(x)$. Third, take a scaling limit. Step 1 is executable analytically, which manifests one of the advantages of the present approach. The resultant equations no longer have dependence on a fictitious N but only depend on the temperature variable, β. After step 2, we obtain the equations, in the Fourier space,

$$\hat{M} \begin{pmatrix} \widehat{\log} Y_{B_1} \\ \widehat{\log} Y_{B_3} \\ \vdots \end{pmatrix} = 4\pi\beta \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} + \hat{K}_0 \begin{pmatrix} \hat{\mathcal{L}}_{B_1} \\ \hat{\mathcal{L}}_{B_3} \\ \vdots \end{pmatrix}$$

where $\hat{\mathcal{L}}_{B_1} = \widehat{\log}\big(1 + \frac{1}{Y_{B_1}}\big)$ and similarly for others. The quantities with a hat indicate that they are Fourier transformations. $\hat{M}$ and $\hat{K}_0$ are asymmetric matrices for which the explicit forms are omitted here but can be easily obtained from the Y-system. The only first entry has a non-vanishing inhomogeneous term in the RHS. This reflects the fact that only Y_{B_1} needs some trivial renormalization so as to have nice analytic properties. By multiplying M^{-1} from the left, the kernel matrix of TBA, $M^{-1}K_0$, turns out to be symmetric, remarkably. This property is crucial in applying the dilogarithm technique to evaluate the central charge. The inhomogeneous term vector $4\pi\beta M^{-1} \cdot {}^t(1, 0, \cdots)$ possesses six non-vanishing elements:

$$\hat{d}_{B_1} = \frac{8\pi\beta \cosh \frac{11}{14}k}{\big(2\cosh \frac{2}{14}k - 1\big)\, D(k)} \qquad \hat{d}_{B_3} = \frac{4\pi\beta \big(2\cosh \frac{2}{14}k + 1\big)\big(2\cosh \frac{4}{14}k - 1\big)}{D(k)}$$

$$\hat{d}_{B_5} = \frac{16\pi\beta \cosh \frac{1}{14}k \cosh \frac{4}{14}k}{\big(2\cosh \frac{2}{14}k - 1\big)\, D(k)} \qquad \hat{d}_{B_2} = \frac{8\pi\beta \cosh \frac{1}{14}k}{\big(2\cosh \frac{2}{14}k - 1\big)\, D(k)}$$

$$\hat{d}_{K_1} = \frac{4\pi\beta}{\big(2\cosh \frac{2}{14}k - 1\big)\, D(k)} \qquad \hat{d}_{K_2} = \frac{8\pi\beta \cosh \frac{4}{14}k}{\big(2\cosh \frac{2}{14}k - 1\big)\, D(k)}$$

where we denote by $\hat{d}_{B_1}$ the drive term associated with $\widehat{\log} Y_{B_1}$ and so on. A common denominator $D(k)$ denotes

$$D(k) = 2\cosh \tfrac{12}{14}k + 2\cosh \tfrac{10}{14}k - 2\cosh \tfrac{6}{14}k - 2\cosh \tfrac{4}{14}k + 1.$$

We finally perform step 3. In view of QFT, the bulk quantity is not of direct interest, rather the fluctuation is. We introduce $y_{B_1}(x) = Y_{B_1}(x + \tau'')$, for example, to evaluate quantities near the 'Fermi surface' with $\tau'' = \frac{12\tau}{7\pi}$. Then take a limit $q \to 0$ such that $m_k R = \frac{8\pi\beta r}{2\cos\frac{\pi}{21} - 1} q^{\frac{4}{7}}$. By r we mean the residue of $\mathrm{i}/D(k)$ at $k = \pi/3\mathrm{i}$. Two quantities M^{-1} and K_0 seem to carry the information of S matrices; the elements of $M^{-1}K_0$ agree with the expression described in [8] in terms of S matrices, under identification $x = 3\theta/\pi$ in the limit $q \to 0$. The matrix M^{-1} also encodes the information of the mass spectra. When taking the inverse Fourier transformation, the nearest zero to the real axis, $k = \pm\mathrm{i}\frac{\pi}{3}$ of $D(k)$, is relevant in the 'scaling' limit as τ'' tends to infinity. Applying the Poisson summation formula, we found a most dominant term

$$d_{K_1}(x) = \frac{8\pi\beta r}{2\cos\frac{\pi}{21} - 1}\, \mathrm{e}^{-\frac{4}{7}\tau} \cosh \frac{\pi}{3}x = m_K R \cosh\theta$$

for example, where $\frac{\pi}{3}x = \theta$. Note that the relation $m_K \propto q^{\frac{4}{7}}$ is consistent with the scaling dimension $\triangle_{1,2} = \frac{1}{8}$. One similarly verifies that all other drive terms also take the form $mR\cosh\theta$ and their mass ratios agree with those in [8]:

$$m_{B_1} = 2m_K \cos\tfrac{11}{42}\pi \qquad m_{B_3} = 4m_K \cos\tfrac{11}{42}\pi \cos\tfrac{3}{42}\pi$$
$$m_{B_5} = 4m_K \cos\tfrac{1}{42}\pi \cos\tfrac{4}{42}\pi \qquad m_{B_2} = 2m_K \cos\tfrac{1}{42}\pi.$$
$$m_{K_2} = 2m_K \cos\tfrac{4}{42}\pi$$

Thus the TBA of $M_{5,6} + \phi_{1,2}$ theory is recovered from the scaling limit of the dilute A_5 model at regime 2.

Once Y is fixed by TBA, we can also evaluate the free energy from

$$T_1\big(x \pm \tfrac{3}{14}\mathrm{i}\big) = T_{B_1}\big(x \pm \tfrac{3}{14}\mathrm{i}\big) = T_0\big(x \pm \tfrac{11}{14}\mathrm{i}\big)(1 + Y_{B_1}(x)).$$

It is readily shown that a 'fluctuation' part of the free energy f is proportional to $\frac{1}{\beta^2}\sum_k \int m_k R\cosh\theta \log(1 + 1/y_k)\,\mathrm{d}\theta$, which is the desired expression.

5. Dilute A_2 model in regime 1 as a lattice analogue to $M_{3,4} + \phi_{2,1}$

We treat another example corresponding to $\phi_{2,1}$ perturbation theory, the simplest and most well-studied case, the Ising model off critical temperature, $M_{3,4}+\phi_{2,1}$. The model is described by a free fermion, thus is rather trivial in a sense. In view of functional relations, however, it is not trivial to derive the simplest Y-system $Y\big(x \pm \mathrm{i}\frac{3}{2}\big) = 1$ (in the present normalization of x), from $T_1(x)$ in (1) which consists of three terms. This model is actually one of the first examples, which require a more fundamental object than $T_1(x)$, a box, which seems to correspond to a fundamental breather B_1.

We define

$$\tau_K(x) := w\phi(x + 2\mathrm{i})\frac{Q(x + 2\mathrm{i})}{Q(x + \mathrm{i})} + \phi(x)\frac{Q(x)Q(x + 3\mathrm{i})}{Q(x + \mathrm{i})Q(x - \mathrm{i})} \tag{9}$$

$$+w^{-1}\phi(x - 2\mathrm{i})\frac{Q(x - 2\mathrm{i})}{Q(x - \mathrm{i})} \tag{10}$$

which has a property common to $T_1(x)$ namely, it is pole-free due to the BAE.

More importantly, we have functional relations,

$$\tau_K\left(x \pm \frac{1}{2}\mathrm{i}\right) = T_1(x) + T_2(x) = 2T_1(x) \tag{11}$$

$$\tau_K\left(x \pm \frac{3}{2}\mathrm{i}\right) = T_3(x) + T_0(x) - \phi\left(x \pm \frac{5}{2}\mathrm{i}\right)\left(w^3 + \frac{1}{w^3}\right)$$
$$= 2\left(\phi\left(x \pm \frac{1}{2}\mathrm{i}\right) + \phi\left(x \pm \frac{5}{2}\mathrm{i}\right)\right). \tag{12}$$

The first equalities are directly verified by comparing both sides in the forms of the ratio of Q functions. The second are consequences of the duality. One then reaches the desired relation (12) after proper renormalizations. The first equation, (11), seems to suggest $\tau_K(x)$ is related to the kink in the theory; the bound state of the kink produces a breather.

In the general $L =$ even case, we find that $\tau_K(x)$ plays the most fundamental role, which will be the topic of a separate publication.

It is a nice exercise to recover from (11) and (12) the free fermion free energy in the scaling limit. We shall note the analytic property, supported by numerics, that $\tau_K(x)$ is analytic and nonzero in the strip $\mathrm{Im}\,x \in \left[-\frac{3}{2}, \frac{3}{2}\right]$, for that purpose.

6. Summary and discussion

In this report, we demonstrate explicitly that TBA for $M_{5,6} + \phi_{1,2}$ and $M_{3,4} + \phi_{2,1}$, conjectured by Dorey *et al*, are realized in the scaling limit of lattice models. The crucial idea is to introduce fusion transfer matrices associated with skew Young tableaux and to investigate the functional relations among them. The proofs of functional relations are rather combinatorial and lengthy, thus omitted due to the lack of space. They will be supplemented in the subsequent paper which discusses the TBA behind the dilute A_L models, L general [29].

There are still many open problems. The explicit identification of string solutions would be definitely one of the most important. The complete study of this will shed some light on the way to proceed for TBA in the case of perturbed non-unitary minimal models. We mention the first step in this direction in [30].

Acknowledgments

The author would like to thank P Dorey and R Tateo for many valuable comments, discussions and collaborations, S O Warnaar for useful correspondence. He also thanks the organizers of RAQIS03 for the nice conference and kind hospitality. This work has been supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports and Technology of Japan, no 14540376.

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